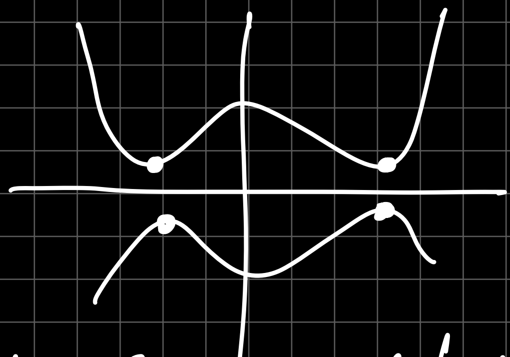
Kane & mele model :-

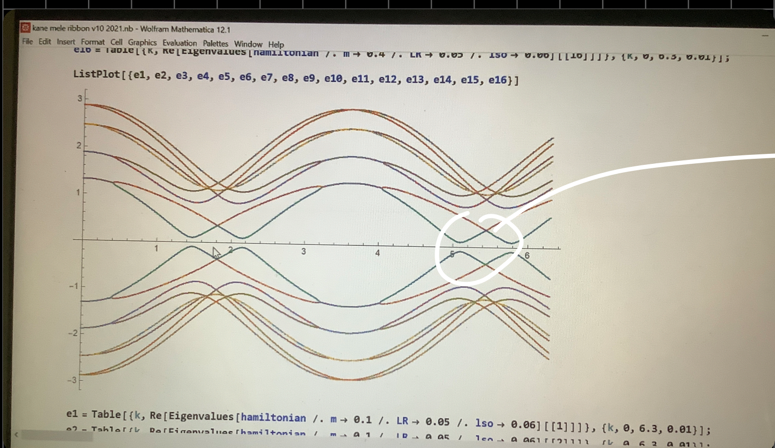
phase :- $\pi/2$, top. non-trivial state

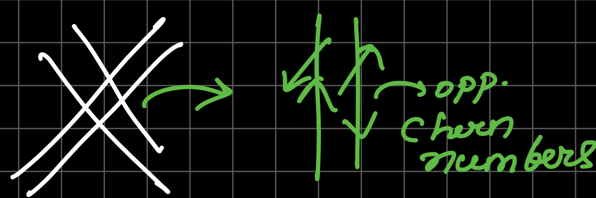
↳ Gap closing , phase term ⇒ makes mass different in 2 valleys



Kane & mele model transport

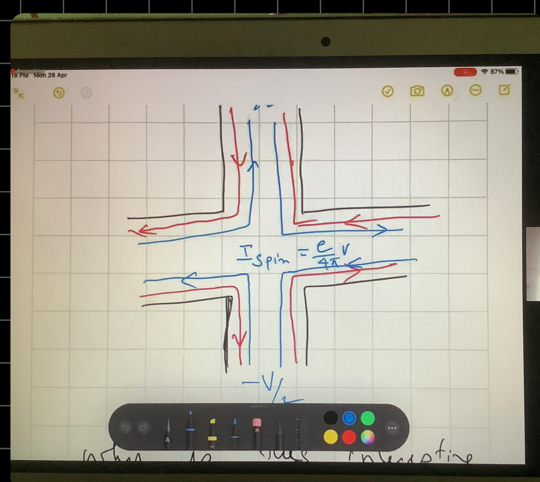
@ mathematica , basis $a\uparrow$ $a\downarrow$ $b\uparrow$ $b\downarrow$ → 4×4





expt differentiate b/w Q Hall & Q Spin Hall state

- Landauer-Buttiker approach



note

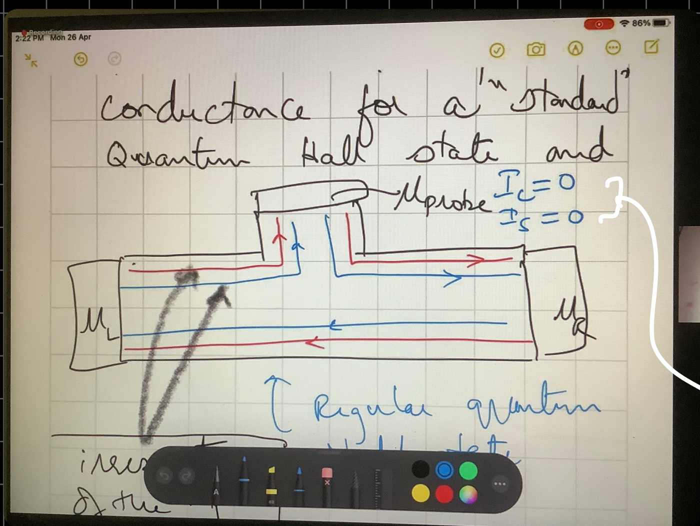
① ballistic transport along the edges with zero magnetic field. Only SOC is requirement

② ∃ a classical spin hall effect - - - - -

origin of this is spin-orbit coupling

Chern insulator states are being explored @ 0K.

• Often a T Geometry is used.

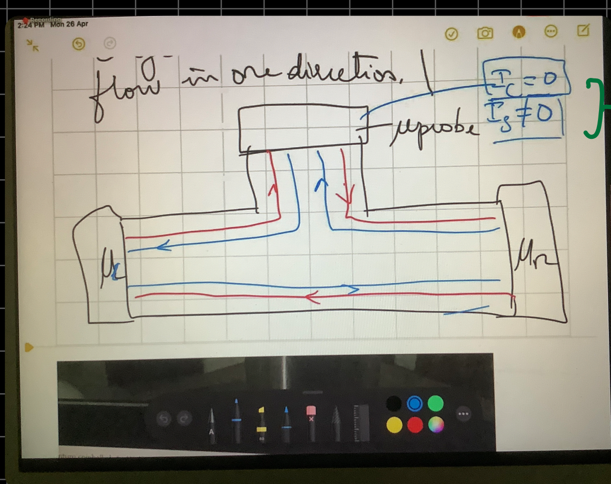


Quantum hall

Book:- make sure μ of all states are same when they leave the probe

floating potential

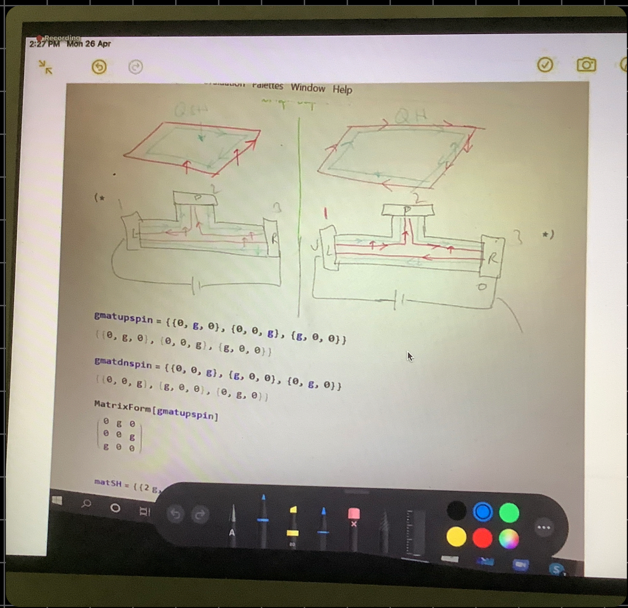
red ↑
blue ↓

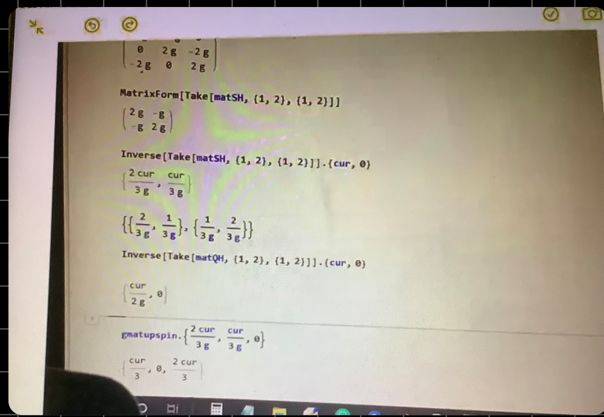


non-zero
net spin current

G matrix → for Landauer buttiker → for $G_\uparrow$ & $G_\downarrow$ &

do it separately.





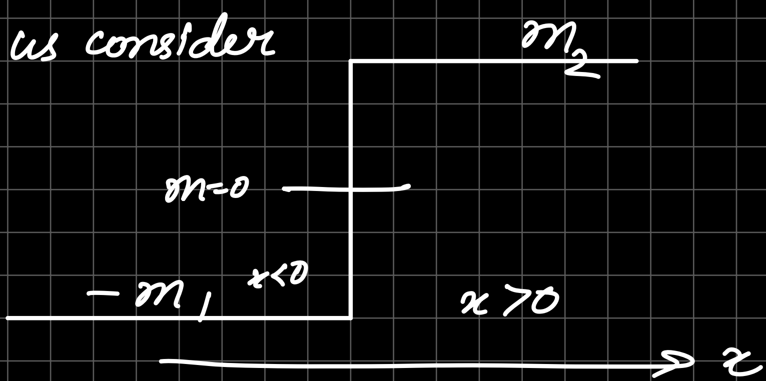
QSHall → has been realised in some realisation of bilayer graphene.

spin current ⇒ no way to directly measure it
↳ however multi-terminal device allow it to be discovered.

Dirac eqn in 1D

$$h(x) = -iv\partial_x \sigma_x + m(x)v^2\sigma_z$$

& let us consider



$$m(x) = \begin{cases} -m_1 & x<0 \\ m_2 & x>0 \end{cases} \qquad \text{with } m_1, m_2 > 0$$

$$\begin{pmatrix} m(x)v^2 & -iv\hbar\partial_x \\ -iv\hbar\partial_x & -m(x)v^2 \end{pmatrix}\begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} = E\begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}$$

we want to look at bound states s.t. ψ vanishes at $\underline{\underline{x = \pm\infty}}$

$$\psi = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} = e^{-\lambda_+ x} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \qquad \underline{\phi}(x) \sim e^{-\lambda_+ x}$$

$$\therefore E\psi = \begin{pmatrix} m(x)v^2 & iv\hbar\lambda \\ iv\hbar\lambda & -mv^2 \end{pmatrix}\psi$$

RHS $\underline{x > 0}$

$$E\psi = \begin{pmatrix} m_2 v^2 & iv\hbar\lambda \\ iv\hbar\lambda & -m_2 v^2 \end{pmatrix}\psi$$

$$E^2 = m^2 v^4 - v^2\hbar^2\lambda^2$$

$$\Rightarrow \lambda = \pm\sqrt{\frac{m_2^2 v^4 - E^2}{v^2\hbar^2}}$$

if $E^2 < m_2^2 v^4$ for a valid sol^n to exist

$$\boxed{\lambda = \lambda_+} > 0$$

if $E^2 > m_2^2 v^4$, then we have a propagating sol^n.

$$\psi = \begin{pmatrix} \varphi_1^+ \\ \varphi_2^+ \end{pmatrix} e^{-\lambda_+ x}$$

LHS $\underline{x < 0}$

$$E\psi = \begin{pmatrix} -m_2 v^2 & iv\hbar\lambda \\ iv\hbar\lambda & +m_2 v^2 \end{pmatrix}\psi$$

$$E^2 = m_2^2 v^2 - v^2\hbar^2\lambda^2$$

$$\Rightarrow \lambda = \pm\sqrt{\frac{m_2^2 v^4 - E^2}{\hbar^2 v^2}}$$

$$\boxed{\psi = e^{-\lambda x}}$$

$x \to -\infty, \psi \to 0$

$\lambda < 0$ for sol^n to be physically relevant

$$\boxed{\therefore \lambda = \lambda_-}$$

$$\psi = \begin{pmatrix} \varphi_1^- \\ \varphi_2^- \end{pmatrix} e^{-\lambda_- x}$$

Now, $\varphi_1^+ = \dfrac{-iv\hbar\lambda_+}{m_2 v^2 - E}\varphi_2^+$ $\qquad$ $\varphi_1^- = \dfrac{-iv\hbar\lambda_-}{-m_1 v^2 - E}\varphi_1^-$

At bdry, $\psi(x \to 0^+) = \psi(x \to 0^-)$

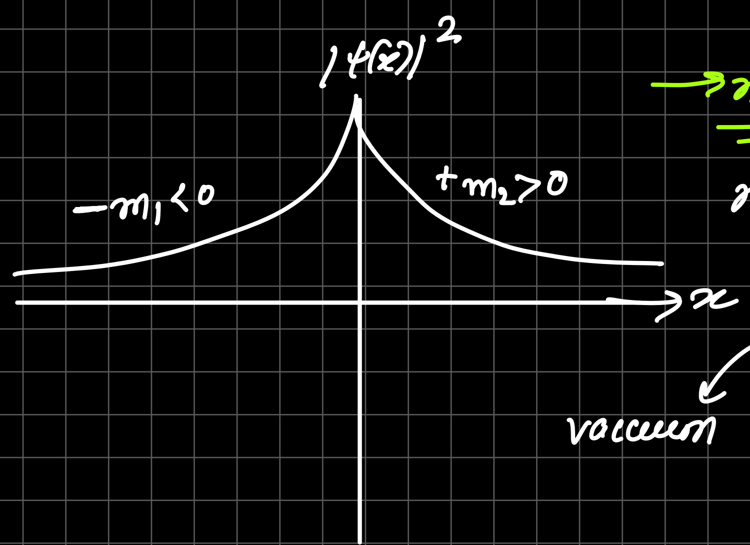
$$\Rightarrow \varphi_1^+ = \varphi_1^-$$

$$\Rightarrow \quad \frac{\sqrt{m_2^2 v^4 - E^2}}{m_2 v^2 - E} = -\frac{\sqrt{m_1^2 v^4 - E^2}}{-m_1 v^2 - E}$$

∴ setting $E=0$ is a valid soln

$$\text{Then} \quad \lambda_+ = +\frac{\sqrt{m_2 v^2}}{v\hbar} = \frac{m_2}{\hbar}, \quad \lambda_- = \frac{-m_1}{\hbar}$$

$$\therefore \text{ net } \psi(x) = \sqrt{\frac{v}{\hbar}\left(\frac{m_1 m_2}{m_1+m_2}\right)} \begin{pmatrix} i \\ 1 \end{pmatrix} e^{-|m(x) v x|/\hbar}$$



→ not diff at $x=0$

mass term

⇓

<u>Conclusion</u> :- If we regard the vacuum as a system with an infinite mass, then any system with a -ve mass (in sense of $m(x)v^2\sigma_z$ term in dirac eqn) along with open BC posses boundary state soln if the 1st derivative continuity condition is <u>relaxed</u>.

Stability :- One can show that for $m(x)$ changing from -ve to +ve across $x=0$, still ∃ a $E=0$ soln across of the <u>Dirac eqn</u>

@ Shen gives the derivation. Net result is

$$\varphi_\eta(x) \sim \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \eta i \end{pmatrix} \exp\left[-\int^x \eta \frac{m(x')v}{\hbar} dx'\right]$$

where $\eta = \pm 1$, determined by signs of $m(\pm\infty)$.

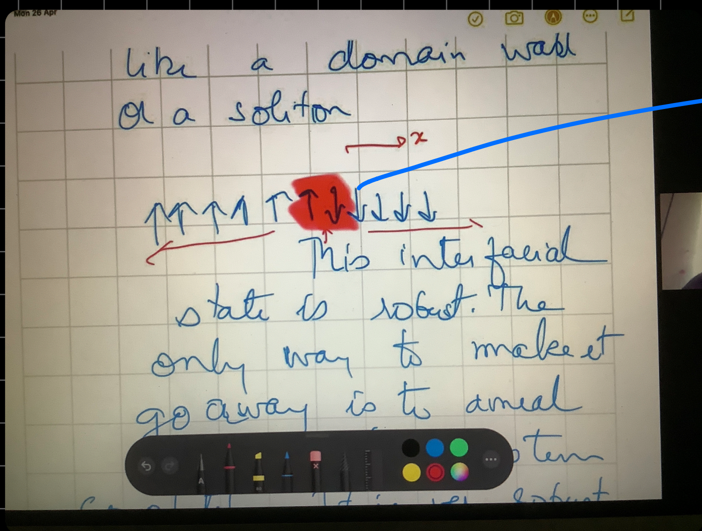
=> "Robust against changing mass distribution"

=> Analogues of such a state

=> Like a domain wall or a soliton

$\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow\downarrow$ → domain wall state

ising domain wall



local adjustment will not get rid of domain wall

robust => local adjustment won't kill it.

→ @ some connection b/w domain wall as a state & the 1d boundary solution

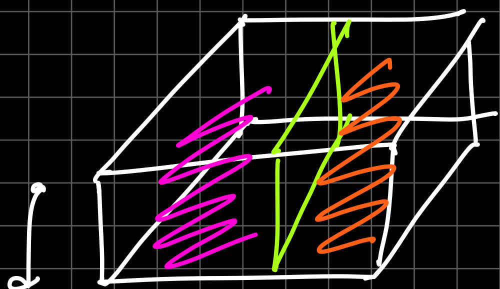
2 Dimensions

$$H = c\,\vec{p}\cdot\vec{\alpha} + mc^2\beta$$

3 Dim. $\alpha_i^2 = \beta^2 = 1 \qquad \alpha_i\alpha_j = -\alpha_j\alpha_i$

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} \sigma_z & 0 \\ 0 & -\sigma_z \end{pmatrix}$$

$$= \sigma_x \otimes \sigma_i$$



very small, so eff 2D

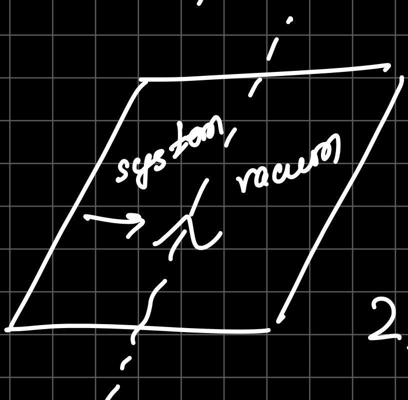
$-m_1 < 0 \quad m_2 > 0$

$\cdot\, p_y = \hbar k_y$ is a good QNo.

Claim:- $\exists$ 2 solutions of 2d Dirac eqn, ($p_z = 0$)

$$H = \underbrace{m(x)v^2\beta + v\,p_x\alpha_x}_{\text{1D part}} + v\,p_y\alpha_y$$

TR symmetric



2 solns:-

$$\psi_+(x,y) = \psi_+^{1D}(x)\, e^{ik_y y} \longrightarrow \text{Trans symm along } y$$

$$\psi_-(x,y) = \psi_-^{1D}(x)\, e^{+ik_y y}$$

Check :- $\left(m(x)v^2\beta + v\alpha_x p_x\right)\psi_\pm = 0$

$$v\,\alpha_y p_y \psi_\pm = v(k_y)(\alpha_y\psi_\pm) = v k_y \sigma_y \psi_\pm = \pm v k_y$$

now

$$\psi_\pm = \sqrt{\frac{v\,m_1 m_2}{\hbar\, m_1 + m_2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix} e^{-|m(x)v x| + i k y} \longrightarrow \text{4 component spinor}$$

with $\epsilon_\pm = \pm v k_y \qquad \therefore\ v_\pm = \pm v$

$i\sigma_y\kappa \rightleftarrows i\sigma_y\kappa \quad \tau = i\sigma_y\kappa \quad \tau^{-1} = -i\sigma_y\kappa$

$\tau^{\dagger} = -\tau$ (anticenitary) we have $\tau|\psi_+\rangle = i\sigma_y\kappa\begin{pmatrix}1\\i\end{pmatrix}$

$$= -i\begin{pmatrix}1\\-i\end{pmatrix}$$

$$= -i|\psi_-\rangle$$

so $\quad \tau|\psi_+\rangle = |\phi\rangle$

$$\langle\phi|\vec{\sigma}|\phi\rangle = \langle\psi_+|\tau^{\dagger}\vec{\sigma}\tau|\psi_+\rangle = \langle\psi_+|-(\vec{\sigma})\quad|\psi_+\rangle$$

$$= -\langle\psi_+|\vec{\sigma}|\psi_+\rangle$$

but $|\phi\rangle = -i|\psi_-\rangle$

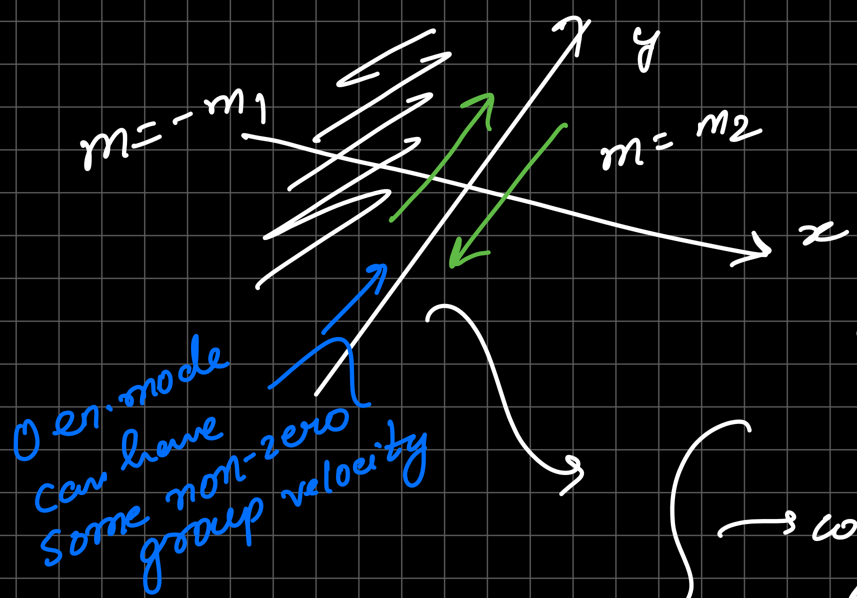
$$\Rightarrow \quad \langle\psi_-|\vec{\sigma}|\psi_-\rangle = -\langle\psi_+|\vec{\sigma}|\psi_+\rangle$$

essentially they're linked by $\sigma_y\kappa$ & hence have opposite spins.

$\therefore$ ψ_+, ψ_- have opposite spins & correspondinly opposite directions of movement.

• As D. eqn is TRS, these constitute **helical edge states.**

for 2D set $p_z = 0$,



2D problem

along y-direction

works for such a boundary as well

i.e. topological property

called helical edge states

$$v_\pm = \frac{\partial \epsilon_{k\pm}}{\hbar \partial k} = \underline{\underline{\pm v}}$$

Why not the Dirac eqn?

① Ambiguity

inconvenience → have to define regions of +ve & −ve mass. (even if $m \to \infty$) & then claim edge states of the −ve massed system.

BUT, $H_{Dirac}(\dots, m, \beta) = H_{Dirac}(\dots, -m, -\beta)$

but $\beta \to -\beta$ is also a valid choice for clifford algebra [i.e. $\{\alpha_i, \alpha_j\} = 0$ & $\{\alpha_i, \beta\} = 0$]

so there's no way to uniquely set +ve (or −ve) mass to a domain (can reverse the choice by a unitary transf.)

\# no topological distinctn b/w system & vacuum.

\# if u use D. eqn ⟶ define an additional "vacuum" as a benchmark.

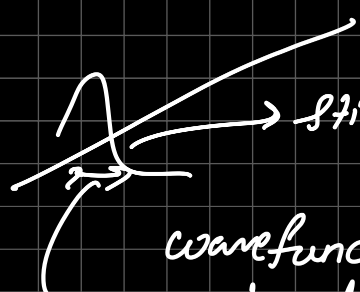
⇓

But bound (edge) state is physical, exists irresp labelling (intrinsic prop of bands texture)

∴ Dirac eqn is itself weak.

②

even if $m_2 \to \infty$

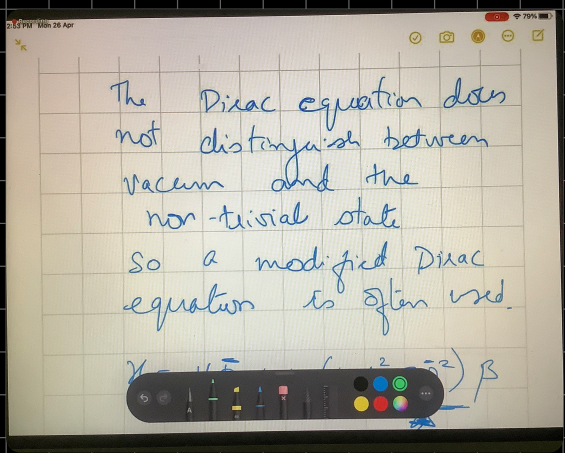


→ still has a finite extent even though $m_2 > 0$

wavefunction exists outside the

material

Need for modified Dirac eqn

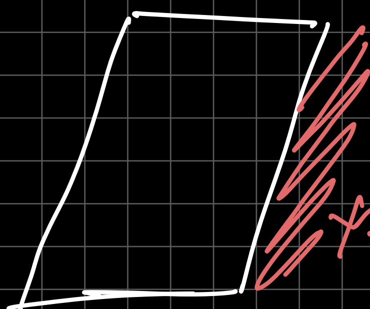

The Dirac equation does not distinguish between vacum and the non-trivial state
So a modified Dirac equation is often used

$$\mathcal{H} = v\,\vec{p}\cdot\vec{\alpha} + (mv^2 - Bp^2)\beta$$

units of m^{-1}

primarily to take care of the mathematics of having a decaying contribution of the vacuum.

breaks symmetry b/w vacuum & the topological system. (i.e. $\mathcal{H}_{Dirac}(\dots, +m, +\beta) \neq \mathcal{H}_{Dirac}(\dots, -m, -\beta)$)



$\psi \to 0$ here

- now $\mathcal{H}_{dirac}$ doesn't leak into vacuum.
- now we don't care about a vacuum.

$$\mathcal{H} = v\,\vec{p}\cdot\vec{\alpha} + \beta(mv^2 - Bp^2)$$

$$E_\pm = \sqrt{v^2p^2 + (mv^2 - Bp^2)^2}$$

What does he mean by spin in GS ???

Spin GS

$$\Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

$$\beta = \begin{pmatrix} \mathbb{1}_{2\times2} & 0 \\ 0 & -\mathbb{1}_{2\times2} \end{pmatrix}$$

rel sign of m, B:-

(1) $mB < 0$ → domain wall structure

$\uparrow\uparrow \leftarrow \uparrow \rightarrow \nearrow\uparrow$ → "Topologically trivial"

$\xleftarrow{\;P\;} \quad \xrightarrow{\;P\;}$

(2) $mB > 0$

spin $\nwarrow \downarrow \uparrow \nwarrow \uparrow \nearrow \nearrow \nearrow \downarrow$

$mB > 0$

"meron"

"topologically non-trivial state"